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Critical Properties and Finite-Size Estimates for the Depinning Transition of Directed Random Polymers

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We consider models of directed random polymers interacting with a defect line, which are known to undergo a pinning/depinning (or localization/delocalization) phase transition. We are interested in critical properties and we prove, in particular, finite-size upper bounds on the order parameter (the *contact fraction*) in a window around the critical point, shrinking with the system size. Moreover, we derive a new inequality relating the free energy F and an annealed exponent μ which describes extreme fluctuations of the polymer in the localized region. For the particular case of a $(1 + 1)$-dimensional interface wetting model, we show that this implies an inequality between the critical exponents which govern the divergence of the disorder-averaged correlation length and of the typical one. Our results are based on the recently proven smoothness property of the depinning transition in presence of quenched disorder and on concentration of measure ideas.

KEY WORDS: directed polymers, pinning and wetting models, copolymers, depinning transition, finite-size estimates, concentration of measure, typical and average correlation lengths

1. INTRODUCTION

Directed polymers interacting with a one-dimensional defect line are quite rich in physical and biological applications, and lately have started to attract much attention also in the mathematical literature.[2,11–15,21] In particular, they are an ideal framework to model $(1 + 1)$-dimensional interface wetting phenomena,[6] the problem of depinning of flux lines from columnar defects in

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type-II superconductors[20] and the denaturation transition of DNA in the Poland-Scheraga approximation.[16] In many situations, the polymer-defect interaction is neither homogeneous nor periodic along the line. This corresponds for instance to the presence of impurities on the wall in the case of the wetting problem, and to the non-periodic arrangement of base pairs A–T, G–C along DNA sequences. Therefore, one resorts very naturally to quenched disordered models.

The interplay between the (*energetic*) pinning effect, which tends to keep the polymer close to the defect line, and the (*entropic*) depinning one, favoring configurations which wander away from the line, is responsible for a non-trivial pinning/depinning (or localization/delocalization) phase transition. The depinned and pinned phases are characterized by a different behavior of the order parameter, the *contact fraction*, which is essentially the density of polymer-defect contacts along the line. In the pinned phase, the contact fraction stays positive in the thermodynamic limit, while it vanishes in the interior of the depinned phase (finite-size estimates of the latter statement can be found in Ref. 12). A very interesting problem is to understand what happens at the *critical line* separating the two phases. Recently, with G. Giacomin we proved that, as soon as disorder is present, the contact fraction vanishes continuously when the critical line is approached from the pinned region.[13] This is in striking contrast with the situation in pure (i.e., non-disordered) pinning models, where the transition can be either of first or of higher order, depending for instance on the space dimension. Given this result, it is very natural to investigate how fast the contact fraction vanishes with system size, *at* the critical line or in a small critical window around it. This question is addressed in Theorem 3.1 of the present paper, where it is shown for instance that, in the disordered situation, the contact fraction is at most of order $N^{-1/3} \log N$ at criticality.

Inside the localized region, the length of the maximal excursion of the polymer (i.e., of the longest portion of the polymer without contacts with the defect line) is $(\log N)/\mu$,[1,15] where μ is a certain annealed exponent (cf. Sec. 3.2 for its definition) and N is the total length of the polymer. When the critical line is approached μ tends to zero, as well as the free energy $\textsc{f}$. In Theorem 3.3 we prove an inequality which essentially relates the critical exponents which govern the vanishing of μ and $\textsc{f}$ at the critical line. This inequality is interesting also because, in the particular case of a $(1+1)$-dimensional wetting model, we prove in Theorem 3.5 that $\textsc{f}^{-1}$ and μ^{-1} coincide with the the typical and disorder averaged correlation lengths of the system, respectively.

As we discuss briefly in Sec. 4, the finite-size estimates of Theorem 3.1 and the bounds of Theorem 3.3 have a very natural generalization to the case of random copolymers at a selective interface between two solvents,[4,11,19] which also show a localization/delocalization transition. In this case, the relevant order parameter is not the contact fraction but the fraction of monomers in the unfavorable solvent.

2. RANDOM PINNING MODELS

Let $S = \{S_n\}_{n=0,1,\ldots}$ be a time-homogeneous process with law $\mathbf{P}$, taking values in some set Σ and such that $S_0 = 0 \in \Sigma$. We will be especially interested in the returns to zero of S: we let $\tau_0 = 0$ and, for $i \geq 1$, $\tau_i = \inf\{j > \tau_{i-1} : S_j = 0\}$. If $\tau_i = \infty$, then by convention $\tau_{i+1} = \infty$. The only assumptions we make on $\mathbf{P}$ is that $\{\tau_i - \tau_{i-1}\}_{i=1,2,\ldots}$ is a sequence of IID random variables taking values in $\mathbb{N} \cup \{\infty\}$ and that, defining $K(n) := \mathbf{P}(\tau_1 = n)$, there exists $s \in \mathbb{N}$ such that

$$K(sn) = \frac{L(n)}{n^\alpha}, \tag{2.1}$$

and $K(n) = 0$ if $n \notin s\mathbb{N}$, for some $1 \leq \alpha < \infty$ and a function $L(\cdot)$ varying slowly at infinity, i.e., a positive function such that $\lim_{x\to\infty} L(xr)/L(x) = 1$ for every $r > 0$.[7] An example of slowly varying function is $r \mapsto (\log(r+1))^b)$, for $b \in \mathbb{R}$, but also $r \mapsto \exp((\log(r+1))^b)$, for $b < 1$, as well as any positive function for which $\lim_{r\to\infty} L(r) > 0$.

On the *defect line* $S \equiv 0$ are placed random charges $\omega = \{\omega_n\}_{n=1,2,\ldots}$ which we assume to be IID *bounded* random variables with law $\mathbb{P}$. We will assume that $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$ (which, as will be clear from (2.2) below, implies no loss of generality). The Hamiltonian describing the interaction between the polymer and the defect line depends on two parameters, $\beta \geq 0$ (playing the role of the strength of the disorder) and $h \in \mathbb{R}$ (where $-h$ represents the average energetic gain of a polymer-line contact):

$$\mathcal{H}_{N,\omega}^{\beta,h}(S) = \sum_{n=1}^{N} (\beta\omega_n - h)\mathbf{1}_{\{S_n=0\}}. \tag{2.2}$$

The corresponding Boltzmann distribution is

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}^{\beta,h}}{\mathrm{d}\mathbf{P}}(S) = \frac{e^{\mathcal{H}_{N,\omega}^{\beta,h}(S)}}{Z_{N,\omega}^{\beta,h}}\mathbf{1}_{\{S_N=0\}} \tag{2.3}$$

and, of course, the partition function is given by

$$Z_{N,\omega}^{\beta,h} = \mathbf{E}\left(e^{\mathcal{H}_{N,\omega}^{\beta,h}(S)}\mathbf{1}_{\{S_N=0\}}\right). \tag{2.4}$$

Here and in the following, we assume that $N \in s\mathbb{N}$, even when not explicitly stated.

As Equation (2.3) shows, the polymer tends to touch the defect line at points where $\beta\omega_n - h > 0$ and to avoid it in the opposite situation. Note that there is a competition between an energetic effect (trying to touch as many favorable points as possible along the line) and an entropic one (trajectories which stay close to the line are much less numerous than those which wander away). Therefore, it is quite intuitive (and actually well known) that a (de) localization transition takes place when the strength of the polymer-line interaction is varied. This will be discussed below.

Remark 2.1. We restrict to bounded disorder variables ω_n just for simplicity of exposition. The results below can be extended to more general situations but we will not pursue this line. Let us just mention that all the results of this paper hold also in the Gaussian case $\omega_1 = \mathcal{N}(0, 1)$. In more general cases of continuous, unbounded disorder variables, a sufficient condition for the results to hold is that the sub-Gaussian concentration inequality (5.2) is satisfied by $\mathbb{P}$ and that a certain condition on the smoothness of the density of ω_1 with respect to the Lebesgue measure on $\mathbb{R}$ holds (cf. Ref. 13, condition **C2**). A discussion of the relevance of concentration of measure inequalities in pinning and copolymer models can be found in Ref. 12.

Remark 2.2. Note that only the model with endpoint S_N pinned to zero is being considered, cf. Eq. (2.3). This is just for simplicity of exposition, since this way one has for $M < N$

$$\log Z_{N,\omega}^{\beta,h} \geq \log Z_{M,\omega}^{\beta,h} + \log Z_{N-M,\theta^M\omega}^{\beta,h} \tag{2.5}$$

(θ is the left shift: $\theta\omega_n = \omega_{n+1}$), a property we will use several times in the proofs of Sec. 5. By the way, note that (2.5) implies that the sequence $\{\mathbb{E} \log Z_{N,\omega}^{\beta,h}\}_N$ is super-additive in N. One could also leave the endpoint free: in this case, in the r.h.s. of Eq. (2.5) error terms of order $\log N$ would appear (cf. e.g. (Ref. 13 Remark 1.1)). As a consequence, in the proof of the theorems one would have to keep track of harmless but annoying logarithmic error terms.

Remark 2.3. To make condition (2.1) more explicit note that, for instance, if $\{S_n\}_n$ is the SRW (simple random walk) on $\Sigma = \mathbb{Z}^d$, then (2.1) holds with $s = 2$ and $\alpha = 3/2$ for $d = 1$ and $\alpha = d/2$ for $d \geq 2$. The Poland-Scheraga model of DNA denaturation also fits into our framework; in this case, the physically relevant value of α is around 2.11.[16] For the Poland-Scheraga model, the contact fraction defined in Eq. (2.8) below corresponds to the fraction of bound base pairs.

As it is well known the infinite-volume free energy, i.e. the limit

$$\textsc{f}(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}^{\beta,h} \tag{2.6}$$

exists, is almost-surely independent of ω and satisfies $\textsc{f}(\beta, h) \geq 0$ (cf. e.g. Ref. 2 and 11, but proofs of these facts have appeared several times in the literature. The non-negativity of $\textsc{f}$ is proven by simply restricting the average in (2.4) to the configurations which do not touch zero between sites 0 and N, and using Eq. (2.1).) One decomposes the phase diagram (β, h) into depinned (or delocalized) and pinned (or localized) phases, $\mathcal{D}$ and $\mathcal{L}$, defined as $\mathcal{D} = \{(\beta, h) : \mathrm{F}(\beta, h) = 0\}$ and $\mathcal{L} = \{(\beta, h) : \mathrm{F}(\beta, h) > 0\}$, separated by a critical line $h_c(\beta) = \inf\{h : \mathrm{F}(\beta, h) = 0\}$. Various properties of the critical curve are known[2,11]: in particular, under our assumptions one has that, for every $0 < \beta < \infty$,

$$h_c(0) = \log(1 - \mathbf{P}(\tau_1 = \infty)) < h_c(\beta) < \infty. \tag{2.7}$$

Note that $h_c(0) \leq 0$, and $h_c(0) < 0$ iff S is transient. Moreover, $h_c(\cdot)$ is a convex increasing function, as follows easily from the convexity of F with respect to its arguments and from (2.7).

The order parameter associated to the (de)localization transition is the *contact fraction*, defined as

$$\ell_N := \frac{\mathcal{N}_N}{N} := \frac{|\{1 \leq n \leq N : S_n = 0\}|}{N}. \tag{2.8}$$

Since F is clearly convex as a function of h, and since it is differentiable in h for every $h < h_c(\beta)$ (as was proven in Ref. 15), from the definitions of $\mathcal{L}, \mathcal{D}$ it follows that, $\mathbb{P}(\mathrm{d}\omega)$-a.s.,

$$\lim_{N\to\infty} \mathbf{E}_{N,\omega}^{\beta,h}(\ell_N) = -\partial_h \mathrm{F}(\beta, h) > 0 \quad \text{if} \quad h < h_c(\beta) \tag{2.9}$$

while

$$\lim_{N\to\infty} \mathbf{E}_{N,\omega}^{\beta,h}(\ell_N) = 0 \quad \text{if} \quad h > h_c(\beta). \tag{2.10}$$

However, much more than (2.10) is true: indeed, in Ref. 12 it was proven that, for $h > h_c(\beta)$,

$$\mathbb{E}\mathbf{P}_{N,\omega}^{\beta,h}(\mathcal{N}_N \geq m) \leq e^{-d_1 m} \tag{2.11}$$

if $m \geq d_2 \log N$, for some constants $0 < d_1(\beta, h), d_2(\beta, h) < \infty$. In other words, the number of contacts with the defect line grows, typically, linearly with N for $h < h_c(\beta)$ and at most logarithmically in N for $h > h_c(\beta)$. Finally, in Refs.13–14 it was proven that $\partial_h \mathrm{F}(\beta, h)$ vanishes continuously for $h \uparrow h_c(\beta)$ if $\beta > 0$, which implies that, $\mathbb{P}(\mathrm{d}\omega)$-a.s.,

$$\lim_{N\to\infty} \mathbf{E}_{N,\omega}^{\beta,h_c(\beta)}(\ell_N) = 0. \tag{2.12}$$

In view of these facts, it is very natural to ask what is the typical size of the contact fraction for finite *N, at* the critical point or very close to it. This question will be addressed in the next section.

3. MAIN RESULTS

3.1. Finite-Size Estimates on the Contact Fraction

Since we are interested in the finite-size scaling behavior of the system in a window around the critical point, shrinking to zero with the system size, we allow in general h to depend on N, and write explicitly $h = h_N$.

Theorem 3.1. *Let $\beta > 0$ and $1 \leq \alpha < \infty$. Assume that*

$$\lim_{N\to\infty} N^t(h_N - h_c(\beta)) = b \in \mathbb{R} \tag{3.1}$$

for some $t \geq 0$. Then,

(1) If $t \geq 1/3$, then for c sufficiently large

$$\lim_{N\to\infty} \mathbb{E}\mathbf{P}_{N,\omega}^{\beta,h_N}\left(\mathcal{N}_N \geq cN^{2/3}\log N\right) = 0 \tag{3.2}$$

(2) If $t < 1/3$ and $b > 0$, then for c sufficiently large

$$\lim_{N\to\infty} \mathbb{E}\mathbf{P}_{N,\omega}^{\beta,h_N}\left(\mathcal{N}_N \geq cN^{2t}\log N\right) = 0. \tag{3.3}$$

(3) If $t < 1/3$ and $b < 0$, then for c sufficiently large

$$\lim_{N\to\infty} \mathbb{E}\mathbf{P}_{N,\omega}^{\beta,h_N}\left(\mathcal{N}_N \geq cN^{1-t}\right) = 0. \tag{3.4}$$

It is understood that the constant c above can depend on β, α and b. Note that, for $t = 0$ and $b > 0$, one finds back the known estimates on the contact fraction valid in the interior of $\mathcal{D}$.[12]

Remark 3.2. The estimates of Theorem 3.1 need not be optimal, in general. Indeed, as will be clear in Sec. 5, our proof is based on the fact that F vanishes at least quadratically when the critical line is approached from the localized region and $\beta > 0$[13].

$$\mathrm{F}(\beta, h) \leq \alpha c_1(\beta)(h_c(\beta) - h)^2 \tag{3.5}$$

for some constant $0 < c_1(\beta) < \infty$, if $h < h_c(\beta)$. On the other hand it is quite reasonable, and actually expected in the physics literature, that the transition is smoother in various situations, for instance if $\alpha \leq 3/2$ and β small. Following the proof of Theorem 3.1 in Sec. 5 it is not difficult to realize (cf. Remark 5.1 below) that, if one assumes

$$\mathrm{F}(\beta, h) \leq c_{\mathrm{F}}(\beta, \alpha)(h_c(\beta) - h)^k \tag{3.6}$$

for every $h < h_c(\beta)$ then, for instance,

$$\lim_{N\to\infty} \mathbb{E}\mathbf{P}^{\beta,h_c(\beta)}_{N,\omega}\left(\mathcal{N}_N \geq cN^{2/(k+1)}\log N\right) = 0, \tag{3.7}$$

for c sufficiently large. If $k > 2$, this would clearly improve the upper bound on the contact fraction at the critical point given by Theorem 3.1. Estimates (3.2)–(3.4) could also be similarly improved for all values of t and b. Unfortunately, up to now there are no known cases where one can prove an estimate like (3.6), with $k > 2$, for non-zero values of β.

3.2. μ, Versus F: An Inequality for Critical Exponents

In Refs. 1 and 15, the quantity

$$\mu(\beta, h) = -\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[\frac{1}{Z^{\beta,h}_{N,\omega}}\right] \tag{3.8}$$

was introduced. As it was proved there, in the localized phase μ is strictly positive and related to maximal excursions of the polymer from the defect line: indeed, for the polymer of length N the maximal distance between two successive returns to zero of S is typically $(\log N)/\mu(\beta, h)$. When h approaches $h_c(\beta)$ from below, μ tends to zero and therefore the length of the maximal excursion diverges, on the scale $\log N$. More precisely, the following bounds were proven in Ref. 15: for every $\beta > 0$ there exists $0 < c_2(\beta) < \infty$ such that

$$c_2(\beta)\textsc{f}(\beta, h)^2 < \mu(\beta, h) < \textsc{f}(\beta, h), \tag{3.9}$$

where the lower bound holds, say, for $0 < h_c(\beta) - h \leq 1$. Our next result significantly improves the lower bound in Eq. (3.9):

Theorem 3.3. *For every $\beta > 0$ there exists* $0 < c_3(\beta) < \infty$ such that

$$0 < -c_3(\beta)\frac{\textsc{f}(\beta, h)^2}{\partial_h\textsc{f}(\beta, h)} < \mu(\beta, h) \tag{3.10}$$

if $0 < h_c(\beta) - h \leq 1$.

Remark 3.4. In order to give a more readable form to these bounds assume that, for $\beta > 0$ and $h < h_c(\beta)$,

$$\textsc{f}(\beta, h) = c_{\textsc{f}}(\beta, (h_c(\beta) - h)^{-1})(h_c(\beta) - h)^{\nu_{\textsc{f}}} \tag{3.11}$$

and

$$\mu(\beta, h) = c_\mu(\beta, (h_c(\beta) - h)^{-1})(h_c(\beta) - h)^{\nu_\mu} \tag{3.12}$$

for some functions $c_{\rm F}(\beta, x)$, $c_\mu(\beta, x)$ slowly varying in x for $x \to \infty$ (of course $\nu_{\rm F}, \nu_\mu \geq 2$, as a consequence of Eq. (3.5) and of the upper bound in Eq. (3.9); in principle, $\nu_{\rm F}, \nu_\mu$ can depend on β). Then, recalling the definition of slow variation and the fact that F is convex in h, one realizes that Eq. (3.9) implies

$$(2 \leq)\nu_{\rm F} \leq \nu_\mu \leq 2\nu_{\rm F}. \tag{3.13}$$

while from (3.10) follows that

$$\nu_\mu \leq \nu_{\rm F} + 1. \tag{3.14}$$

3.3. Typical and Average Correlation Lengths for a $(1+1)$-Dimensional Wetting Model

Beyond giving informations about the divergence of the longest excursion close to (but below) the critical line, bounds like (3.10) involving μ and F are of interest because it is rather natural to expect that μ^{-1} (respectively F^{-1}) has the same divergence, for h approaching $h_c(\beta)$ from the localized phase, as the average (respectively typical) correlation length of the system. Our next result, Theorem 3.5, makes this conjecture precise at least in a specific model of $(1+1)$-dimensional wetting.

Recall that in (Ref. 15, Theorem 2.2) it was proven that, for every bounded local observable A (i.e., bounded function which depends on S_j only for j in a finite subset of $\mathbb{N}$), the infinite-volume limit

$$\mathbf{E}^{\beta,h}_{\infty,\omega}(A) = \lim_{N\to\infty} \mathbf{E}^{\beta,h}_{N,\omega}(A) \tag{3.15}$$

exists $\mathbb{P}(d\omega)$-almost surely, if $(\beta, h) \in \mathcal{L}$. Moreover, in $\mathcal{L}$ truncated correlation functions decay exponentially fast with distance. In fact, for every bounded local observables A, B define the local observable B_k as $B_k(S) = B(\theta^k S)$, where θ is the left shift, $\theta S_n = S_{n+1}$. Then, there exist a constant $0 < C_{A,B}(\beta, h) < \infty$, an almost surely finite random variable $C_{A,B}(\omega, \beta, h)$ and a constant $d(\beta, h) > 0$ such that,[15] in $\mathcal{L}$,

$$\mathbb{E}\left|\mathbf{E}^{\beta,h}_{\infty,\omega}(AB_k) - \mathbf{E}^{\beta,h}_{\infty,\omega}(A)\mathbf{E}^{\beta,h}_{\infty,\omega}(B_k)\right| \leq c_{A,B}e^{-d(\beta,h)k} \tag{3.16}$$

and

$$\left|\mathbf{E}^{\beta,h}_{\infty,\omega}(AB_k) - \mathbf{E}^{\beta,h}_{\infty,\omega}(A)\mathbf{E}^{\beta,h}_{\infty,\omega}(B_k)\right| \leq c_{A,B}(\omega)e^{-d(\beta,h)k}. \tag{3.17}$$

However, in Ref. 15 the (β, h) dependence of the constant $d(\beta, h)$ was not tracked, and lower bounds complementary to Eqs. (3.16) and (3.17) were not obtained. It turns out that this gap can be filled, at least in the case of a rather natural $(1+1)$-dimensional wetting model we define now. This model still belongs to the class described by the Boltzmann distribution (2.3) but, in addition to the basic assumptions of Sec. 2, we require that the state space of the process S is $\Sigma = \mathbb{Z}^+$ (i.e., there is an impenetrable wall which prevents $S_n < 0$) and that actually S is the SRW with increments $S_{i+1} - S_i = \pm 1$, conditioned to be non-negative (the condition $|S_i - S_{i-1}| = 1$ could be somewhat relaxed in the theorem below, at the price of some further technical work. We will not pursue this line). Note that in this case (2.1) holds with $\alpha = 3/2$ and $s = 2$. This model has a natural interpretation as a (1+1)-*dimensional wetting model of a disordered substrate*.(2,6,9) The defect line represents a wall with impurities, and S the interface between two coexisting phases (say, liquid below the interface and vapor above). When $h < 0$ the underlying homogeneous substrate repels the liquid phase, and vice versa for $h > 0$. $\mathcal{L}$ corresponds then to the *dry phase* (microscopic liquid layer at the wall) and $\mathcal{D}$ to the *wet phase* (macroscopic layer).

Then, one has:

Theorem 3.5. *For the wetting model just introduced, the following holds: for every* $\beta \geq 0$ *and* $h < h_c(\beta)$,

$$-\lim_{k\to\infty} \frac{1}{k} \log \mathbb{E}\left(\mathbf{P}^{\beta,h}_{\infty,\omega}(S_\ell = S_{\ell+k} = 0) - \mathbf{P}^{\beta,h}_{\infty,\omega}(S_\ell = 0)\mathbf{P}^{\beta,h}_{\infty,\omega}(S_{\ell+k} = 0)\right) = \mu(\beta, h) \tag{3.18}$$

and, $\mathbb{P}(d\omega) - a.s.$,

$$-\lim_{k\to\infty} \frac{1}{k} \log \mathbb{E}\left(\mathbf{P}^{\beta,h}_{\infty,\omega}(S_\ell = S_{\ell+k} = 0) - \mathbf{P}^{\beta,h}_{\infty,\omega}(S_\ell = 0)\mathbf{P}^{\beta,h}_{\infty,\omega}(S_{\ell+k} = 0)\right) = \mathrm{F}(\beta, h). \tag{3.19}$$

Here it is understood that $\ell, k, N \in 2\mathbb{N}$, due to the periodicity of the simple random walk.

Remark 3.6. It would be extremely interesting, especially in view of Theorem 3.5, to fill the gap between the upper bound in Eq. (3.9) and the lower bound (3.10) (or equivalently, between (3.13) and (3.14)). In the case of the $(1 + 1)$-dimensional wetting model with ± 1 increments, this would answer the question whether typical and average correlation lengths have the same critical behavior close to the depinning transition, or if their divergence is governed by different critical exponents, as it happens for instance in the disordered Ising spin chain with random transverse field of Ref. 8.

4. GENERALIZATION TO COPOLYMERS AT A SELECTIVE INTERFACE

In this Section we sketch briefly how Theorems 3.1 and 3.3 can be extended to the model of *random copolymer at a selective interface*.(4,11,19) We refer for instance to[11,22] for physical motivations of this model. In this case, the state space of S is $\Sigma \equiv \mathbb{Z}$ and, in addition to time homogeneity of S and to the IID property of the sequence $\{\tau_i - \tau_{i-1}\}_i$, one assumes that $(S_{i+1} - S_i) \in \{-1, 0, +1\}$ and that $\mathbf{P}$ is invariant under the transformation $S \to -S$. The Hamiltonian (2.2) is replaced by

$$\hat{\mathcal{H}}^{\beta,h}_{N,\omega}(S) = \sum_{n=1}^{N} (\beta\omega_n - h)\mathbf{1}_{\{S_n<0\}} \tag{4.1}$$

where, without loss of generality in view of the symmetry of $\mathbf{P}$, we can assume that $h \geq 0$. The variables $\{\omega_n\}_n$ are IID centered and satisfy the same boundedness assumption as in Sec. 2. The Boltzmann distribution and the partition function $\hat{Z}^{\beta,h}_{N,\omega}$ are defined as in Eqs. (2.3) and (2.4), provided that $\mathcal{H}^{\beta,h}_{N,\omega}$ is replaced by $\hat{\mathcal{H}}^{\beta,h}_{N,\omega}$. One should imagine the model as describing a polymer S in proximity of the interface ($S \equiv 0$) between two solvents A and B, placed in the half-planes $S > 0$ and $S < 0$, respectively. Note that S_n has the tendency to be in A whenever $\beta\omega_n - h < 0$ and in B if $\beta\omega_n - h > 0$. Note also that, if $h > 0$, for a typical disorder realization the polymer has a net preference to be in A, which will be called the *favorable solvent.*

Again, it is known[4] that the infinite-volume free energy $\hat{\mathrm{F}}(\beta, h) = \lim_N (1/N) \log \hat{Z}^{\beta,h}_{N,\omega}$ exists, is almost surely independent of ω and non-negative, so that one can define the localized and delocalized phases, $\hat{\mathcal{L}}$ and $\hat{\mathcal{D}}$, is analogy to Sec. 2. Upper[4] and lower[3] bounds are known for the critical curve $h_c(\beta) = \inf\{h : \hat{\mathrm{F}}(\beta, h) = 0\}$ but, on the basis of careful numerical simulations plus concentration of measure considerations, none of them is believed to be optimal in general.[5] In contrast with the case of the pinning models of Sec. 2, for the copolymer the order parameter associated to the localization/delocalization transition is the *fraction of monomers in the unfavorable solvent*:

$$\hat{\ell}_N := \frac{\hat{\mathcal{N}}_N}{N} := \frac{|\{1 \leq n \leq N : S_n < 0\}|}{N}. \tag{4.2}$$

This is rather intuitive since, comparing definitions (2.2) and (4.1), one notices that the role of $\mathbf{1}_{\{S_n=0\}}$ is now played by $\mathbf{1}_{S_n<0}$. Like for the contact fraction in pinning models, various estimates on the order parameter are known: $\hat{\ell}_N$ is of order 1 in $\hat{\mathcal{L}}$, at most of order $(\log N)/N$ in the interior of $\hat{\mathcal{D}}$[12] and $o(1)$ for $N \to \infty$ at the critical line.[13] The methods we introduce in the present paper allow to make the last statement sharper: indeed, Theorem 3.1 holds unchanged also for the copolymer model, provided that $\mathcal{N}_N$ is replaced by $\hat{\mathcal{N}}_N$. In particular, therefore, $\hat{\ell}_N$ is at most of order $N^{-1/3} \log N$ at the critical point.

Theorem 3.3 also admits a natural extension to the copolymer case: if $\hat{\mu}(\beta, h)$ is defined as in (3.8), with $Z^{\beta,h}_{N,\omega}$ replaced by $\hat{Z}^{\beta,h}_{N,\omega}$, then again Eq. (3.10) holds with F, μ replaced by $\hat{\mathrm{F}}$, $\hat{\mu}$.

In order to avoid a useless duplication of the proofs of Theorems 3.1 and 3.3, in Sec. 5 we will consider only the case of pinning models and we will not give details for the copolymer case: as it was also the case in Refs. 12–15, it is easy to realize that the two models can be treated analogously, if the correct order parameter is used in each case. Just to give an example, Eq. (5.4) below holds also for the copolymer, if $\mathcal{N}_N$ is replaced by $\hat{\mathcal{N}}_N$, as was proven in (Ref. 12, Lemma 2.1).

5. PROOF OF THE RESULTS

Given a set Ω of polymer configurations, measurable with respect to **P**, it is convenient to set

$$Z_{N,\omega}^{\beta,h}(\Omega) := \mathbf{E}\left(e^{\mathcal{H}_{N,\omega}^{\beta,h}(S)}\mathbf{1}_{\{S\in\Omega\}}\mathbf{1}_{\{S_N=0\}}\right). \tag{5.1}$$

Our basic technical tool is the following classical concentration inequality[17]: if $\omega = \{\omega_n\}_n$ is a sequence of IID bounded random variables with law $\mathbb{P}$, there exist constants $0 < C_1, C_2 < \infty$ such that, for every convex Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, one has:

$$\mathbb{P}(|f(\omega_1, \ldots, \omega_n) - \mathbb{E}f(\omega_1, \ldots, \omega_n)| \geq t) \leq C_1 \exp\left(-\frac{C_2 t^2}{\|f\|_{Lip}^2}\right) \tag{5.2}$$

for every $t > 0$, where $\|f\|_{Lip}$ is the Lipschitz norm of f with respect to the Euclidean norm in $\mathbb{R}^n$, i.e., the smallest $M \geq 0$ such that

$$\sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|f(x) - f(y)|}{[\Sigma_{i=1}^n (x_i - y_i)^2]^{1/2}} \leq M. \tag{5.3}$$

The way we will employ this inequality is by noting that $(1/N)\log Z_{N,\omega}^{\beta,h}$, considered as a function of $\omega_1, \ldots, \omega_N$, is convex and has a Lipschitz constant at most $\beta/\sqrt{N}$. More generally, one has the following (Ref. 12, Lemma 2.1): let Ω_m be a set of polymer trajectories such that $\mathcal{N}_N \leq m$ for every $S \in \Omega_m$. Then,

$$\mathbb{E}\left(\left|\frac{1}{N}\log Z_{N,\omega}^{\beta,h}(\Omega_m) - \frac{1}{N}\mathbb{E}\log Z_{N,\omega}^{\beta,h}(\Omega_m)\right| \geq t\right) \leq C_1 \exp\left(-C_2\frac{N^2t^2}{\beta^2 m}\right). \tag{5.4}$$

This is simply proven by noting that $(1/N)\log Z_{N,\omega}^{\beta,h}(\Omega_m)$ has a Lipschitz constant at most $\beta\sqrt{m}/N$.

Proof of Theorem 3.1. For $m \in \mathbb{N} \cup \{0\}$, consider the restricted partition function

$$Z_{N,\omega}^{\beta,h}(\mathcal{N}_N = m) \tag{5.5}$$

where the number of contacts with the line, $\mathcal{N}_N$, is constrained to m. Thanks to the fact that the differences $\tau_i - \tau_{i-1}$ between successive return times to zero of S are independent under the law $\mathbf{P}$, one has

$$\frac{1}{N}\mathbb{E}\log Z_{N,\omega}^{\beta,h}(\mathcal{N}_N = m) \leq \lim_{k\to\infty}\frac{1}{kN}\mathbb{E}\log Z_{kN,\omega}^{\beta,h}(\mathcal{N}_{kN} = km)$$
$$\leq \phi\left(\beta, \frac{m}{N}\right) - h\frac{m}{N}, \tag{5.6}$$

where

$$\phi(\beta, x) = \lim_{\varepsilon\searrow 0}\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_{N,\omega}^{\beta,0}(\ell_N \in [x-\varepsilon, x+\varepsilon]). \tag{5.7}$$

The limits in Eq. (5.7) exist for monotonicity reasons. In Ref. 13 it was proven that, under some assumptions on $\mathbb{P}$ (assumptions which are satisfied, in particular, in the case of bounded random variables we are considering here), one has for $\beta > 0$

$$\phi(\beta, x) \leq -\frac{c_4(\beta)}{\alpha}x^2 + h_c(\beta)x, \tag{5.8}$$

for some constant $0 < c_4(\beta) < \infty$ depending only on the law $\mathbb{P}$.

Remark 5.1. Equation (5.8) follows simply from Eq. (3.5) and from the fact that, as was proven in Ref. 13, F is related to the function ϕ of Eq. (5.7) via the Legendre transformation

$$\mathrm{F}(\beta, h) = \sup_{x\in[0,1]}(\phi(\beta, x) - hx). \tag{5.9}$$

(Actually, in Ref. 13 the reverse path was followed: first (5.8) was proven, and then (3.5) was deduced). If one could prove Eq. (3.6) with $k > 2$, (5.8) would be immediately improved into

$$\phi(\beta, x) \leq -\widetilde{c}_{\mathrm{F}}(\beta, \alpha)x^{k/(k-1)} + h_c(\beta)x \tag{5.10}$$

for some $0 < \widetilde{c}_{\mathrm{F}}(\beta, \alpha) < \infty$.

Equation (5.8), together with (5.6), implies that for every $N \in s\mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$

$$\frac{1}{N}\mathbb{E}\log Z_{N,\omega}^{\beta,h_N}(\mathcal{N}_N = m) \leq -\frac{c_4(\beta)}{\alpha}\left(\frac{m}{N}\right)^2 - (h_N - h_c(\beta))\frac{m}{N}. \tag{5.11}$$

Let us consider first the case $b > 0, t < 1/3$. Then, for N sufficiently large one has, uniformly in m,

$$\frac{1}{N}\mathbb{E}\log Z_{N,\omega}^{\beta,h_N}(\mathcal{N}_N = m) \leq -\frac{b}{2}N^{-t}\frac{m}{N}. \tag{5.12}$$

We let E_1 be the event

$$E_1 = \left\{ \text{there exists } m \geq cN^{2t} \log N \text{such that } \frac{1}{N} \log Z_{N,\omega}^{\beta,h_N}(\mathcal{N}_N = m) \right.$$
$$\left. \geq -\frac{b}{4} N^{-t} \frac{m}{N} \right\}. \tag{5.13}$$

To estimate the probability of E_1, we employ Eq. (5.4) and we find

$$\mathbb{P}[E_1] \leq C_1 \sum_{m \geq cN^{2t} \log N} e^{-C_2 \frac{b^2 m N^{-2t}}{16\beta^2}} \tag{5.14}$$

which decays to zero for $N \to \infty$, if c is large enough. On the complementary of the event E_1, one the other hand, one has

$$\mathbf{P}_{N,\omega}^{\beta,h_N}(\mathcal{N}_N \geq cN^{2t} \log N) = \frac{\sum_{m \geq cN^{2t} \log N} Z_{N,\omega}^{\beta,h_N}(\mathcal{N}_N = m)}{Z_{N,\omega}^{\beta,h_N}}$$
$$\leq c_5 N^{2\alpha} \sum_{m \geq cN^{2t} \log N} e^{-\frac{bmN^{-t}}{4}} \tag{5.15}$$

which also decays to zero. In Eq. (5.15) we used the obvious bound

$$Z_{N,\omega}^{\beta,h} \geq Z_{N,\omega}^{\beta,h}(\{S_n \neq 0 \text{ for every } n < N\}) = K(N) e^{\beta\omega_N - h} \geq (c_5)^{-1} N^{-2\alpha}, \tag{5.16}$$

cf. Eq. (2.1) and the definition of slowly varying function. Equations (5.14) and (5.15) together imply (3.3).

Next, consider the case $t \geq 1/3$. It is immediate to check that, for N sufficiently large and $m \geq cN^{2/3} \log N$, the r.h.s. of Eq. (5.11) is smaller than

$$-\frac{c_4(\beta)}{2\alpha} \left(\frac{m}{N}\right)^2.$$

Then, one defines

$$E_2 = \left\{ \text{there exists } m \geq cN^{2/3} \ \log\ N \text{ such that } \frac{1}{N} \log Z_{N,\omega}^{\beta,h_N}(\mathcal{N}_N = m) \right.$$
$$\left. \geq -\frac{c_4(\beta)}{4\alpha} \left(\frac{m}{N}\right)^2 \right\} \tag{5.17}$$

and notes that, in analogy with Eqs. (5.14) and (5.15),

$$\mathbb{P}[E_2] \leq C_1 \sum_{m \geq cN^{2/3} \log N} e^{-C_2 \frac{c_4(\beta)^2 m^3}{16\alpha^2 N^2 \beta^2}} \tag{5.18}$$

while, on the complementary of the event E_2,

$$\mathbf{P}^{\beta,h_N}_{N,\omega}(\mathcal{N}_N \geq cN^{2/3}\log N) \leq c_5 N^{2\alpha} \sum_{m \geq cN^{2/3}\log N} e^{-\frac{c_4(\beta)m^2}{4N}}, \tag{5.19}$$

which together imply (3.2), for c large.

Finally, the case $b < 0$ and $t < 1/3$. One realizes easily that, for N, c sufficiently large and $m \geq cN^{1-t}$, the r.h.s. of Eq. (5.11) is smaller than

$$-|b|N^{-t}\frac{m}{N}.$$

Then, one defines

$$E_3 = \left\{\exists m \geq cN^{1-t} \text{ such that } \frac{1}{N}\log Z^{\beta,h_N}_{N,\omega}(\mathcal{N}_N = m) \geq -\frac{|b|}{2}N^{-t}\frac{m}{N}\right\} \tag{5.20}$$

and notes that

$$\mathbb{P}[E_3] \leq C_1 \sum_{m \geq cN^{1-t}} e^{-C_2\frac{b^2}{4\beta^2}N^{-2t}m} \tag{5.21}$$

which decays to zero for $N \to \infty$ since $t < 1/3$ while, on the complementary of the event E_3,

$$\mathbf{P}^{\beta,h_N}_{N,\omega}(\mathcal{N}_N \geq cN^{1-t}) \leq c_5 N^{2\alpha} \sum_{m \geq cN^{1-t}} e^{-\frac{|b|}{2}N^{-t}m}. \tag{5.22}$$

Equation (3.4) follows as in the previous cases. □

Proof of Theorem 3.3. Define preliminarily, for every $x \in [0, 1]$ and $\varepsilon > 0$,

$$E_{N,x,\varepsilon} := \left\{\omega : \frac{1}{N}\log Z^{\beta,h}_{N,\omega}(\ell_N \in [x-\varepsilon, x+\varepsilon]) < \frac{\textsc{f}(\beta,h)}{2}\right\}. \tag{5.23}$$

Then,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{Z^{\beta,h}_{N,\omega}}\right] &\leq \exp(-N\textsc{f}(\beta,h)/2) + \mathbb{E}\left[\frac{\mathbf{1}_{\{E_{N,x,\varepsilon}\}}}{Z^{\beta,h}_{N,\omega}}\right] \\ &\leq \exp(-N\textsc{f}(\beta,h)/2) + c_5 N^{2\alpha}\mathbb{P}[E_{N,x,\varepsilon}].\end{aligned} \tag{5.24}$$

Thanks to the Legendre transformation relation (5.9) and from the infinite differentiability of the free energy for $h < h_c(\beta)$,[15] it follows that the value $\bar{x}(h)$, which realizes the supremum in Eq. (5.9), is unique, smooth as a function of h and satisfies $\bar{x}(h) = -\partial_h \textsc{f}(\beta,h)$. Moreover, since $\phi(\beta,\bar{x}(h)) - h\bar{x}(h) = \textsc{f}(\beta,h)$, one has immediately

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z^{\beta,h}_{N,\omega}(\ell_N \in [\bar{x}(h)-\varepsilon, \bar{x}(h)+\varepsilon]) = \textsc{f}(\beta,h). \tag{5.25}$$

Thanks to Eq. (5.4), one has then for ε sufficiently small

$$\mathbb{P}\left[E_{N,\bar{x}(h),\varepsilon}\right] \leq C_1 e^{-C_2 \frac{N\mathrm{F}(\beta,h)^2}{8\beta^2(-\partial_h \mathrm{F}(\beta,h))}} \tag{5.26}$$

for N sufficiently large. Therefore, recalling Eq. (5.24), always for N large one finds

$$\mathbb{E}\left[\frac{1}{Z_{N,\omega}^{\beta,h}}\right] \leq \exp(-N\mathrm{F}(\beta,h)/2) + C_1 e^{-C_2 \frac{N\mathrm{F}(\beta,h)^2}{16\beta^2(-\partial_h \mathrm{F}(\beta,h))}} \tag{5.27}$$

which immediately implies Eq. (3.10) for $h_c(\beta) - h > 0$ sufficiently small. Indeed, since $\mathrm{F}(\beta, .)$ is a convex function and $\mathrm{F}(\beta, h_c(\beta)) = 0$, one has

$$\frac{\mathrm{F}(\beta,h)}{-\partial_h \mathrm{F}(\beta,h)} \leq h_c(\beta) - h,$$

which implies that, for $h_c(\beta) - h$ small, the second term in the r.h.s. of Eq. (5.27) is the larger one.

Proof of Theorem 3.5. Recall that here $\ell, k, N \in 2\mathbb{N}$. We start with the upper bounds on the correlation lengths, which are somewhat easier. Observe first that

$$\begin{aligned} C_{N,\omega}^{\beta,h}(\ell,k) &:= \mathbf{P}_{N,\omega}^{\beta,h}(S_\ell = S_{\ell+k} = 0) - \mathbf{P}_{N,\omega}^{\beta,h}(S_\ell = 0)\mathbf{P}_{N,\omega}^{\beta,h}(S_{\ell+k} = 0) \\ &= \mathbf{E}_{N,\omega}^{\beta,h,\otimes 2}\left[\left(\mathbf{1}_{\{S_\ell^1 = S_{\ell+k}^1 = 0\}} - \mathbf{1}_{\{S_\ell^1 = S_{\ell+k}^2 = 0\}}\right)\mathbf{1}_{\{E\}}\right], \end{aligned} \tag{5.28}$$

where $\mathbf{P}_{N,\omega}^{\beta,h,\otimes 2}(\cdot)$ is the product Gibbs measure for two independent, identical copies S^1, S^2 of the polymer and E is the event

$$E = \left\{\nexists\, j : \ell < j < \ell + k,\, S_j^1 = S_j^2\right\}. \tag{5.29}$$

Indeed, the expectation in Eq. (5.28) vanishes if conditioned on the complementary of E, as is immediately realized via a symmetry argument based on the Markov property of the SRW conditioned to be non-negative. An analogous trick was used in the proof of (Ref. 15, Theorem 2.2). Then, it follows that

$$\begin{aligned} C_{N,\omega}^{\beta,h}(\ell,k) &\leq \mathbf{P}_{N,\omega}^{\beta,h,\otimes 2}(E) = 2\mathbf{P}_{N,\omega}^{\beta,h,\otimes 2}\left(S_j^2 > S_j^1 \forall j : \ell < j < \ell + k\right) \\ &\leq 2\mathbf{P}_{N,\omega}^{\beta,h}(S_j > 0\ \forall j : \ell < j < \ell + k) \end{aligned} \tag{5.30}$$

where in the second and third steps we used the fact that, since the polymer trajectories have increments of unit length, S^1 and S^2 cannot cross without touching. At this point, let us condition on the last return to zero of S before $\ell + 1$, which we call m, and on its first return r after $\ell + k - 1$, and observe that

$$Z_{N,\omega}^{\beta,h} \geq Z_{N,\omega}^{\beta,h}(S_m = S_r = 0) = Z_{m,\omega}^{\beta,h} Z_{r-m,\theta^m\omega}^{\beta,h} Z_{N-r,\theta^r\omega}^{\beta,h} \tag{5.31}$$

where, we recall, θ is the left shift: $\theta\omega_n = \omega_{n+1}$. From (5.30) one obtains

$$\begin{aligned} C^{\beta,h}_{N,\omega}(\ell,k) &\le 2 \sum_{\substack{0\le m<\ell\\ \ell+k\le r\le N}} \mathbf{P}^{\beta,h}_{N,\omega}(\{S_m = S_r = 0\}\cap\{S_j > 0 \forall j : m < j < r\}) \\ &\le 2 \sum_{\substack{0\le m<\ell\\ \ell+k\le r\le N}} \frac{K(r-m)e^{\beta\omega_r - h}}{Z^{\beta,h}_{r-m,\theta^m\omega}} \le c_6 \sum_{\substack{0\le m\le\ell\\ \ell+k\le r}} \frac{1}{Z^{\beta,h}_{r-m,\theta^m\omega}}. \end{aligned} \tag{5.32}$$

Recalling the definition (3.8) of μ, and the fact that $(1/s)\log Z^{\beta,h}_{s,\omega}$ converges to $\textsc{f}(\beta,h)$ $\mathbb{P}(\mathrm{d}\omega)$-a.s. for $s\to\infty$, one obtains for every $\delta > 0$

$$\mathbb{E}C^{\beta,h}_{N,\omega}(\ell,k) \le c_7 e^{-(\mu(\beta,h)-\delta)k} \tag{5.33}$$

and

$$C^{\beta,h}_{N,\omega}(\ell,k) \le c_8(\omega)e^{-(\mathrm{F}(\beta,h)-\delta)k} \tag{5.34}$$

where $c_8(\omega)$ is $\mathbb{P}(\mathrm{d}\omega)$-almost surely finite. Here and in the following we omit the possible dependence on β, h and ℓ of the constants, in order to keep notations lighter. Note however that c_7 can be chosen independent of ℓ. Since neither c_7 nor $c_8(\omega)$ depend on N, the $N\to\infty$ limit can be taken in the l.h.s. of Eqs. (5.33) and (5.34).

As for the lower bound, we start by observing that, by Eq. (5.28), one has the identity

$$C^{\beta,h}_{N,\omega}(\ell,k) = \mathbf{P}^{\beta,h,\otimes 2}_{N,\omega}\left(\left\{S^1_\ell = S^1_{\ell+k} = 0\right\}\cap\left\{S^2_j > S^1_j \ \forall j : \ell\le j\le \ell+k\right\}\right). \tag{5.35}$$

Indeed, thanks to the constraint $\mathbf{1}_{\{E\}}$, it cannot happen that $S^1_\ell = S^2_\ell = 0$, otherwise also $S^1_{\ell+1} = S^2_{\ell+1} = 0$, since $S_j \ge 0$ and $|S_j - S_{j-1}| = 1$. Similarly, it cannot happen that $S^1_{\ell+k} = S^2_{\ell+k} = 0$. For this reason, the first term in the last line of (5.28) gives the r.h.s. of (5.35). In view of analogous considerations, the second term is identically zero, since there are no polymer configurations belonging to E, i.e., not crossing each other, and satisfying $S^1_\ell = S^2_{\ell+k} = 0$. On the other hand, thanks to (Ref. 15, Lemma A.1), one can bound

$$Z^{\beta,h}_{N,\omega} \le c_9 k^{c_9} Z^{\beta,h}_{N,\omega}(S_i = S_j = 0) \tag{5.36}$$

for some c_9 independent of ω, provided that $i, j \le 2k$. Indeed, Lemma A.1 of Ref. 15 states that there exists an ω-independent constant $0 < c_{10} < \infty$ such that for every $N, k\in 2\mathbb{N}$, $k\le N$ and every ω we have

$$\mathbf{P}^{\beta,h}_{N,\omega}(S_k = 0) \ge \frac{1}{c_{10}(k\wedge(N-k))^{c_{10}}} e^{-\beta|\omega_k|-h}, \tag{5.37}$$

from which inequality (5.36) easily follows.

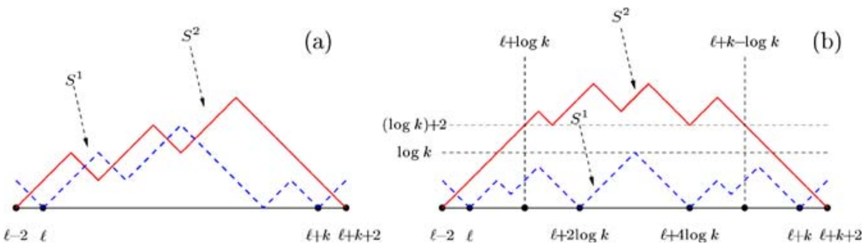

Fig. 1. (a): Typical trajectories $S^1 \in A_1^{\ell,k}$ (dashed line) and $S^2 \in A_2^{\ell,k}$ (full line). (b): Typical trajectories $S^1 \in A_3^{\ell,k}$ (dashed line) and $S^2 \in A_4^{\ell,k}$ (full line). We assumed to simplify the picture that $\log k$ is an integer number and that k is multiple of $\log k$. S^2 is constrained to go up with slope 1 between $\ell - 2$ and $\ell + \log k$, and to go down with slope -1 between $\ell + k - \log k$ and $\ell + k + 2$. Between $\ell + \log k$ and $\ell + k - \log k$, S^2 cannot go below level $\log k + 2 = S^2_{\ell+\log k}$. Therefore, S^2 never touches zero between $\ell - 1$ and $\ell + k + 1$ and S^1 is strictly lower than S^2 between ℓ and $\ell + k$.

In order to keep notations in the following formulas simple, let us introduce some useful sets of polymer trajectories (see Figure 1):

$$
\begin{aligned}
A_1^{\ell,k} &:= \{S : S_\ell = S_{\ell+k} = 0\} \\
A_2^{\ell,k} &:= \{S : S_{\ell-2} = S_{\ell+k+2} = 0\} \\
A_3^{\ell,k} &:= \left\{S \in A_1^{\ell,k} : S_{\ell+j\lfloor \log k\rfloor} = 0 \text{ for every } j \in 2\mathbb{N},\, 1 \le j \le \left\lfloor \frac{k}{\lfloor \log k\rfloor} \right\rfloor \right\} \qquad (5.38)\\
A_4^{\ell,k} &:= \big\{S \in A_2^{\ell,k} : S_{\ell+\lfloor \log k\rfloor} = S_{\ell+k-\lfloor \log k\rfloor} = \lfloor \log k\rfloor + 2 \text{ and} \\
&\qquad S_j > \lfloor \log k\rfloor + 1 \text{ for } \ell + \lfloor \log k\rfloor \le j \le \ell + k - \lfloor \log k\rfloor\big\}.
\end{aligned}
$$

Of course, $A_4^{\ell,k}$ is non-empty only for k sufficiently large so that $k \ge 2\lfloor \log k\rfloor$. If $\hat\Omega$ is a $\mathbf{P}^{\otimes 2}$-measurable set of trajectories of S^1, S^2 we define, in analogy with Eq. (5.1),

$$
Z_{N,\omega}^{\beta,h,\otimes 2}(\hat\Omega) := \mathbf{E}^{\otimes 2}\left(e^{\mathcal{H}_{N,\omega}^{\beta,h}(S^1)+\mathcal{H}_{N,\omega}^{\beta,h}(S^2)} \mathbf{1}_{\{(S^1,S^2)\in\hat\Omega\}} \mathbf{1}_{\{S^1_N=0\}} \mathbf{1}_{\{S^2_N=0\}}\right). \qquad (5.39)
$$

Then, one has the obvious lower bound

$$
C_{N,\omega}^{\beta,h}(\ell,k) \ge \frac{Z_{N,\omega}^{\beta,h,\otimes 2}\Big(\big\{S^1 \in A_1^{\ell,k}\big\} \cap \big\{S_j^2 > S_j^1 \forall j : \ell \le j \le \ell + k\big\} \cap \big\{S^2 \in A_2^{\ell,k}\big\}\Big)}{\left(Z_{N,\omega}^{\beta,h}\right)^2} \qquad (5.40)
$$

and, thanks to Eq. (5.36),

$$
\left(Z_{N,\omega}^{\beta,h}\right)^2 \le c_9^2 k^{2c_9} Z_{N,\omega}^{\beta,h}\big(S \in A_1^{\ell,k}\big) Z_{N,\omega}^{\beta,h}\big(S \in A_2^{\ell,k}\big). \qquad (5.41)
$$

The numerator in Eq. (5.40) can be bounded below requiring that $S^1 \in A_3^{\ell,k}$ and $S^2 \in A_4^{\ell,k}$. At this point the constraint $\{S_j^2 > S_j^1 \forall j : \ell \le j \le \ell + k\}$ becomes superfluous, since it is automatically satisfied if $S^1 \in A_3^{\ell,k}$ and $S^2 \in A_4^{\ell,k}$, and one obtains

$$C_{N,\omega}^{\beta,h}(\ell,k) \geq c_9^{-2} k^{-2c_9} \frac{Z_{k,\theta^\ell\omega}^{\beta,h}\big(S \in A_3^{0,k}\big)}{Z_{k,\theta^\ell\omega}^{\beta,h}} \frac{Z_{k+4,\theta^{\ell-2}\omega}^{\beta,h}\big(S \in A_4^{2,k}\big)}{Z_{k+4,\theta^{\ell-2}\omega}^{\beta,h}}. \tag{5.42}$$

Note that the trajectories belonging to $A_4^{2,k}$ never touch the defect line in the interval $\{1,\ldots,k+3\}$. Therefore, in $Z_{k+4,\theta^{\ell-2}\omega}^{\beta,h}(S \in A_4^{2k})$ the pinning Hamiltonian gives no contribution except at the boundary point k, and one is left with a SRW computation. An easy counting of allowed trajectories gives, for large k,

$$Z_{k+4,\theta^{\ell-2}\omega}^{\beta,h}\big(S \in A_4^{2,k}\big) \geq k^{-c_{11}} \tag{5.43}$$

uniformly in ω. Secondly, applying repeatedly (Ref. 15, Lemma A.1) one obtains

$$Z_{k,\theta^\ell\omega}^{\beta,h}\big(S \in A_3^{0,k}\big) \geq c_{12}^{-k/\log k}(\log k)^{-c_{12}k/\log k} Z_{k,\theta^\ell\omega}^{\beta,h}. \tag{5.44}$$

Plugging the lower bounds (5.43), (5.44) into (5.42) and taking the $N \to \infty$ limit one finally finds

$$C_{\infty,\omega}^{\beta,h}(\ell,k) \geq \frac{c_{13} e^{-c_{14}\frac{k}{\log k}\log(\log k)}}{Z_{k+4,\theta^\ell\omega}^{\beta,h}}. \tag{5.45}$$

The conclusions

$$\mathbb{E} C_{\infty,\omega}^{\beta,h}(\ell,k) \geq c_{15} e^{-(\mu(\beta,h)+\delta)k} \tag{5.46}$$

and

$$C_{\infty,\omega}^{\beta,h}(\ell,k) \geq c_{16}(\omega) e^{-(\mathrm{F}(\beta,h)+\delta)k} \tag{5.47}$$

are obtained, for every $\delta > 0$, by recalling the definition of $\mu(\beta,h)$ and the fact that $(1/k)\log Z_{k,\omega}^{\beta,h}$ converges to $\mathrm{F}(\beta,h)$ almost surely. Together with Eqs. (5.33) and (5.34), these imply the desired results (3.18), (3.19).

Remark 5.2. It is interesting to compare the strategy leading to the upper bounds (5.33), (5.34) with the coupling method introduced in Ref. 18 to estimate the speed of convergence to equilibrium of some special renewal sequences. The connection between polymer measures and renewal equations is not casual: for instance, a moment of reflection (or a look at Appendix A of Ref. 12) shows that, in the homogeneous case $\beta = 0$, the polymer measure can be rewritten exactly in terms of the renewal process where the probability that the time elapsed between two successive renewals is n is given by $K(n)\exp(-n\mathrm{F}(0,h) - h)$.

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